European Journal of Operational Research 114 (1999) 219–233

EUROPEAN JOURNAL OF OPERATIONAL RESEARCH

Heuristic algorithms for the portfolio selection problem with minimum transaction lots

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Received 18 December 1996; accepted 25 November 1997

Abstract

The problem of selecting a portfolio has been largely faced in terms of minimizing the risk, given the return. While the complexity of the quadratic programming model due to Markowitz has been overcome by the recent progress in algorithmic research, the introduction of linear risk functions has given rise to the interest in solving portfolio selection problems with real constraints. In this paper we deal with the portfolio problem with minimum transaction lots. We show that in this case the problem of finding a feasible solution is, independently of the risk function, NP-complete. Moreover, given the mixed integer linear model, new heuristics are proposed which starting from the solution of the relaxed problem allow to find a solution close to the optimal one. The algorithms are based on the construction of mixed integer subproblems (using only a part of the securities available) formulated using the information obtained from the solution of the relaxed problem. The heuristics have been tested with respect to two disjoint time periods, using real data from the Milan Stock Exchange.

Keywords: Integer programming; Heuristics; Portfolio optimization; Minimum transaction lots

1. Introduction

The portfolio selection problem has found a first mathematical formulation in the pioneering paper of Markowitz [1] thanks to which the investments diversification has been translated into computational terms. In the last 40 years we have been witness to a great evolution with respect to the traditional Mean–Variance (MV) scheme, introduced by Markowitz. Some of the main drawbacks recognized to MV model are its high computational complexity and the input problem of estimating $2n + n(n-1)/2$ parameters (expected returns, variances and covariances), which made the model a milestone in finance theory, but a scarcely used tool in practice. This situation justified the several attempts in literature to linearize the quadratic objective function (see [2–4]).

Nowadays MV models consisting of more than a few thousand assets have been solved changing dramatically the practical role of MV approach for constructing large scale portfolios. Real time solutions are obtainable through the use of interior

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point algorithm for quadratic programming problem [5], or by using compact factorizations and piecewise linear approximations [6,7].

The first linear model for portfolio selection is due to Konno and Yamazaki [8]. The linear form of the model is made possible by the use of a risk function different from the classical portfolio variance, namely the portfolio absolute deviation. A relevant feature of the model is that no probabilistic assumptions are made on the securities rates of return, while in the case the rates of return are multivariate normally distributed the model is shown to be equivalent to Markowitz's one. The Konno and Yamazaki's model, the so-called Mean Absolute Deviation (MAD), has been applied by Zenios and Kang [9] to a mortgage-backed securities portfolio optimization in which the rates of return distribution is asymmetric. Speranza [10] introduced a more general model with a weighted risk function. The author showed how a suitable choice for the coefficients in the linear combination gives rise to a model equivalent to Konno and Yamazaki's but halving its number of constraints. A similar result has been independently obtained by Feinstein and Thapa [11].

The largest part of the portfolio selection models which have been proposed in the literature are based on the assumption of a perfect fractionability of the investments in such a way that the portfolio fraction for each security could be represented by a real variable. In the real world, securities are negotiated as multiples of a minimum transaction lot (the so called rounds). As a consequence of considering rounds, solving a portfolio selection problem requires finding the solution of a mixed integer programming model. When applied to real problems, the tractability of the integer model is subject to the availability of algorithms able to find a good, even if not optimal, integer solution in a reasonable amount of time. A general mixed integer model including real characteristics of the problem has been presented in Speranza [12], where a simple heuristic is proposed and tested for the case when minimum transaction lots are considered. The problem with fixed transaction costs with and without minimum transaction lots has been recently studied in Mansini and Speranza [13].

In this paper we show that, when rounds are taken into account, the problem of finding a feasible solution is, independently of the risk function, NP-complete. Moreover, new algorithms are proposed for the solution of the model with rounds. As the number of securities selected by a standard (quadratic or linear) portfolio optimization model is observed to be almost always smaller than 20, the heuristics proposed herein are based upon the idea of constructing and solving mixed integer subproblems which consider subsets of the investment choices available. The subsets are generated by exploiting the information obtained from the relaxed problem (selected securities and reduced costs). The heuristics have the relevant advantage of being general. Different mixed integer models can be of interest in portfolio selection if, for instance, transaction costs are considered. The presented algorithms can be applied or easily generalized to such models.

We present three different heuristics based on the above idea, each of which represents a dominating version of the previous one. The solutions obtained using data from the Milan Stock Exchange with respect to the time periods (1989–1991) and (1992–1994) show that the errors generated by the first and the second heuristic are always less than 5.588% and than 2.615%, with respect to the optimal solution value, respectively. The errors found by the third heuristic, computed with respect to the continuous relaxation solution value, are always smaller than 1.472%.

The most effective of the heuristics gives a maximum error of 1.472% over the period (1989–1991) and of 1.323% over the period (1992–1994), both errors computed with respect to the continuous relaxed optimum; besides, it finds the optimal solution on about 65% and 80% of the tested instances of the two time intervals, respectively. Moreover, even the simplest of the heuristics outperforms the heuristic proposed in [12].

The paper is organized as follows. In Section 2 the general Mean Semi-absolute Deviation model including minimum transaction lots is described, after a brief overview of Markowitz's classical portfolio selection problem and of Konno and Yamazaki's linear model. NP-completeness results for the Portfolio Feasibility Problem with and without tight bounds on the capital are given in Section 3. In Section 4 the new heuristics are presented and in Section 5 the experiments and the results on data from the Milan Stock Exchange are discussed. Finally, some remarks on future research are made.

2. The Mean Semi-absolute Deviation model with minimum lot constraints

Markowitz's original work was based on the rule that the investor does consider expected return as a desirable thing and variance as an undesirable one. Analytically, this implies that given $|S|$ securities, where S is the set of investment alternatives (securities), and a level ρ of expected return the model turns out to be a quadratic programming problem as follows:

$$\min \sum_{i\in S}\sum_{j\in S}\sigma_{ij}x_i x_j, \tag{1}$$

$$\sum_{i\in S} r_i x_i = \rho, \tag{2}$$

$$\sum_{i\in S} x_i = 1, \tag{3}$$

$$x_i \geqslant 0, \quad i \in S, \tag{4}$$

where x_i represents the percentage of money invested in security i, $r_i = E(R_i)$ with R_i the random variable representing the return of security i and σ_{ij} is the covariance between returns of security i and of security j. The most commonly adopted assumption for this model is multivariate normally distributed rates of return.

In 1991 Konno and Yamazaki [8] developed a new approach having an important implication in portfolio analysis especially when the previous assumptions are not satisfied. In their original formulation of the L_1 risk model, Konno and Yamazaki proposed the following risk function:

$$\min \quad w(x) = E\left[\left|\sum_{j\in S} R_j x_j - E\left[\sum_{j\in S} R_j x_j\right]\right|\right]. \tag{5}$$

The random variable R_j still represents the rate of return, while x_j is the amount of money invested in security j.

According to Konno and Yamazaki, r_{jt} is the realization of the random variable R_j during the period t and is obtainable through historical (or forecasted) data. Alternative models in which different scenarios for the rates of returns are taken into account are described in [10]. In particular, they assume that the mean of R_j can be estimated as

$$r_j = E[R_j] = \frac{\sum_{t=1}^{T} r_{jt}}{T}, \tag{6}$$

where T is the length of the time horizon, and that $w(x)$ can be reformulated as follows:

$$w(x) = \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{j\in S}(r_{jt} - r_j)x_j\right|. \tag{7}$$

This objective function is equivalent to the following linear program:

$$\min \frac{\sum_{t=1}^{T} y_t}{T} \tag{8}$$

$$y_t + \sum_{j\in S}(r_{jt} - r_j)x_j \geqslant 0, \quad t = 1,\ldots,T, \tag{9}$$

$$y_t - \sum_{j\in S}(r_{jt} - r_j)x_j \geqslant 0, \quad t = 1,\ldots,T. \tag{10}$$

If the risk is measured by means of the mean semi-absolute deviation instead of the absolute deviation, as in Speranza [10], the objective function is

$$\frac{\sum_{t=1}^{T} |\min\{0, \sum_{j\in S}(r_{jt} - r_j)x_j\}|}{T} \tag{11}$$

and can be rewritten as

$$\min \frac{\sum_{t=1}^{T} y_t}{T}, \tag{12}$$

$$y_t + \sum_{j\in S}(r_{jt} - r_j)x_j \geqslant 0, \quad t = 1,\ldots,T, \tag{13}$$

that is with a smaller number of constraints. It has been shown in [10] that (11) is equivalent to (7) and thus to the variance under the assumption of multivariate normally distributed returns.

Since the model based on a mean semi-absolute deviation risk function is linear, it becomes natural to introduce new specifications deriving from market structure as well as from operative constraints.

We briefly describe the required notation for the mixed integer model with minimum transaction lot constraints. We denote by c_j the purchasing price for the minimum lot of security j. In this way, for each security, the minimum lot is expressed in terms of money and is equivalent to $c_j = N_j p_j$, where p_j is the market price for security j at the date of the purchase of the portfolio and N_j is the number of units of security j required as minimum quantity. Trivially, it is $c_j = p_j$ when asset j is traded without minimum lot.

The integer variable x_j, $\forall j \in S$, represents the number of minimum lots, for each security j, which will make part of the optimal portfolio. The quantity $c_j x_j$ indicates the part of the total available amount of money that the investor decides to put in security j. For example, the Exchange Board establishes that security 5 has to be bought in multiples of 5000 units (i.e. $N_5 = 5000$). On the date we decide to purchase a portfolio, security 5 has a price equal to $p_5 = 2700$ Italian Liras. This implies that security 5 has a minimum round of 13.5 million Italian Liras (i.e. $c_5 = 13{,}500{,}000$). If the final solution shows that $x_5 = 7$, then an amount of 94.5 million Italian Liras of the total fund C will be invested in security 5.

The constant d_j, which may vary according to market conditions and agreement types, expresses the transaction cost proportional to the value of the purchase. In Section 5 an example of how the d_j are computed is given. Since the proportional transaction costs can be directly incorporated in the price, from now on we assume that the price c_j includes all possible proportional transaction costs.

The mixed integer linear program for the portfolio selection problem with minimum lot constraints is

$$\min \frac{\sum_{t=1}^{T} y_t}{T}, \tag{14}$$

$$y_t + \sum_{j \in S} (r_{jt} - r_j) c_j x_j \geqslant 0, \quad t = 1, \ldots, T, \tag{15}$$

$$C = \sum_{j \in S} c_j x_j, \tag{16}$$

$$\sum_{j \in S} r_j c_j x_j \geqslant \rho C, \tag{17}$$

$$C_0 \leqslant C \leqslant C_1, \tag{18}$$

$$0 \leqslant x_j \leqslant u_j, \quad \text{integer } j \in S, \tag{19}$$

$$y_t \geqslant 0, \quad t = 1, \ldots, T. \tag{20}$$

Constraint (16) defines as C the total portfolio expenditure. The constraint on the expected return (17) implies that the selected portfolio has a combined rate of return greater than ρ. With constraint (18), the unknown investment C is fixed to range between C_0 and C_1, i.e. between the minimum and the maximum amount of money available for the investment. Finally, constraints (19) define limitations on the value each x_j can take, being $c_j u_j$ an upper bound on the investment in security j.

3. NP-completeness results

The portfolio selection problem with minimum transaction lots establishes the minimization of a risk function $f(x)$ given constraints on the return and on the investment and it can be formulated as follows:

$$\min f(x),$$

$$\sum_{j \in S} c_j x_j = C,$$

$$\sum_{j \in S} r_j c_j x_j \geqslant \rho C,$$

$$C_0 \leqslant C \leqslant C_1,$$

$$0 \leqslant x_j \leqslant u_j, \quad \text{integer } j \in S,$$

where u_j represents an upper bound on the number of lots purchased of security j.

We will refer to the problem of identifying a feasible solution of the above problem as the Portfolio Feasibility Problem. We now show that the Portfolio Feasibility Problem is NP-complete both in the case with $C_0 = C_1$ and in the case with $C_0 \neq C_1$.

Theorem 1. *If $C_0 = C_1$ the Portfolio Feasibility Problem is NP-complete.*

Proof. We show a transformation from the Partition Problem to the Portfolio Feasibility Problem. The Partition Problem, which is known to be NP-complete (see Garey and Johnson [14]), is defined as follows:

Instance: A set $A = \{a_1, a_2, \ldots, a_n\}$ with $a_i \geqslant 0$ and such that $\sum_{i=1}^{n} a_i = 2s$.

Question: Does there exist a partition of A into two subsets A_1 and A_2 such that

$$\sum_{a_i \in A_1} a_i = \sum_{a_i \in A_2} a_i = s?$$

Let an instance of the Partition Problem be given. We associate to it an instance of the Portfolio Feasibility Problem with $|S| = n$. For each item a_i, we create a security i whose unit cost is a_i and such that $u_i = 1$ is the upper bound on the number of units x_i. The capital bounds are $C_0 = C_1 = s$, the expected rate of return is $\rho = 0$ and the r_{it} are arbitrarily chosen in such a way that $r_i \geqslant 0$, $\forall i$.

If the Partition Problem has "yes" answer let $x_i = 1$ if $a_i \in A_1$ and $x_i = 0$ otherwise. Then the portfolio obtained by selecting x_i units of security i satisfies the budget constraint. Moreover, as $a_i \geqslant 0$ and $\rho = 0$ the constraint on the expected rate of return is satisfied as well. Similarly, it can be easily seen that if a feasible solution of the Portfolio Problem exists, then by including in A_1 the items such that $x_i = 1$ the Partition Problem has "yes" answer. □

Theorem 2. *If $C_0 \neq C_1$ the Portfolio Feasibility Problem is NP-complete.*

Proof. We show a transformation to the Portfolio Feasibility Problem from a particular case of the Knapsack Problem, known to be NP-complete (see Garey and Johnson [14]) and defined as:

Instance: A set $A = \{a_1, a_2, \ldots, a_n\}$, with $a_i \geqslant 0$, and positive integers B and K.

Question: Is there a subset $A' \subseteq A$ such that $\sum_{a_i \in A'} a_i \leqslant B$ and such that $\sum_{a_i \in A'} a_i \geqslant K$?

Take an instance of the special case of the Knapsack Problem. We associate to such an instance, an instance of the Portfolio Feasibility Problem with $|S| = n$. For each item a_i, we create a security i whose unit cost is a_i and such that $u_i = 1$ is the upper bound on the number of units x_i. The capital bounds are $C_0 = K$ and $C_1 = B$, the expected rate of return is $\rho = 0$ and the r_{it} are arbitrarily chosen in such a way that $r_i \geqslant 0$, $\forall i$.

It can be easily verified that the Portfolio Feasibility Problem has "yes" answer if and only if the special case of the Knapsack Problem has "yes" answer. □

Although the Portfolio Feasibility Problem is NP-complete both when $C_0 = C_1$ and when $C_0 \neq C_1$, it is clear that a feasible solution of the portfolio selection problem with minimum lots is more likely to exist when the gap between C_0 and C_1 is not too tight.

4. Linear programming based heuristics

Aiming at facing the complexity of a MILP, it is necessary to work out some heuristic methods which approximate the optimal integer solutions when the instance of the problem is too large to be optimally solved with an efficient commercial package, such as CPLEX. A heuristic method for the above problem can be found in Speranza [12], where an algorithm inspired by the natural behavior of an investor is presented. The algorithm is based on the solution of the LP-relaxation and a subsequent adjustment of the continuous solution which has the aim of reaching feasibility with respect to the original constraints. The algorithm has the advantage of being simple, but the main disadvantage of having been designed for a specific problem. A modification of the above model with the introduction of additional features of the real problem makes the design of a new algorithm necessary. On the contrary, the heuristics presented in the following have a general scheme which can be applied to modifications of the above model.

4.1. Procedure A: Basic MILP-based heuristic

The heuristic is based on solving the continuous relaxation of the problem and retaining the securities with a positive value in the solution. Then the original problem is solved on this subset of securities.

The procedure is essentially composed of the following steps.

1. *Relaxed problem solution*: Solve the LP-relaxation of the problem. Let us denote by x_R the vector of solutions and by n the number of x_R components with value greater than zero (i.e. n is the number of securities selected by the relaxed problem), while $|S| - n$ represents the remaining null components.

2. *Local MILP construction*: Construct a mixed integer linear programming problem only using the n positive components of x_R. This problem has $n + T$ variables and a total number of constraints equal to $T + 2$.

3. *Local MILP solution*: Solve the mixed integer problem. Let x_I be the vector of solutions.

An evident pitfall for this procedure is that the limited number of securities considered to formulate the new instance at step 2 can exclude some desirable securities. The following algorithms will overcome the problem by using a different criterion of selection.

4.2. Procedure B: Reduced cost MILP-based heuristic

The present procedure is an extension of the previous one: the number of variables for the local MILP is no more limited to the positive components of the relaxed problem solution. The local MILP instance is constructed using the first k (where k is established from the beginning) securities selected after sorting the variables according to the nondecreasing order of the reduced costs. The procedure is organized as follows.

1. *Relaxed problem solution*: Solve the relaxed problem. Let x_R be the solutions vector.

2. *Sort on securities*: Sort the securities according to the nondecreasing order of the reduced cost coefficients. We assume that x_R is now the ordered solutions vector. Let n be the number of positions of x_R filled in by securities with positive values and null reduced costs, while, in the remaining $|S| - n$ positions, the securities have value equal to zero and positive reduced costs sorted in nondecreasing order.

3. *Local MILP construction*: Select the first k securities (with $k > n$), in order to construct the local MILP problem. k can be determined by using one of the following rules

(a) Fix k as

$$k = n + pn, \tag{21}$$

where p represents a convenient percentage.

(b) Take k equal to the number of constraints in the original problem, i.e. take $k = T + 2$. The rule takes into account the fact that in the relaxed problem the maximum number of variables with a value different from zero is exactly equal to the number of the constraints.

(c) Fix k to a value, independent from n, which is the maximum number of variables for which the MILP problem can be solved within a reasonable amount of time and space.

4. *Local MILP solution*: Solve the MILP problem. Let x_I be the vector of solutions.

Since k is greater than the number n of positive components in the relaxed problem, this solution dominates the solution of the procedure A.

It is worth noticing that each rule for the value of k in step 3 gives rise to a variant of the procedure. Moreover, a different formulation of the heuristic can be easily obtained by replacing step 2 with a step in which the securities are sorted according to a different criterion, e.g. by the nondecreasing order of the absolute deviation normalized with respect to the expected return.

4.3. Procedure C: Iterated MILP-based heuristic

As a further extension of heuristic B the following algorithm uses an iterative procedure which changes at each iteration the securities on which the mixed integer model is solved.

1. *Relaxed problem solution*: Solve the relaxed problem. Let x_R be the solutions vector.

2. *Sort on securities*: Sort the securities according to the nondecreasing order of the reduced cost coefficients.

3. *Securities selection*: Select the first k securities of vector x_R using one of the three rules presented in the description of heuristic B. Cancel the first k components of the vector x_R.

4. *Local MILP problem construction*: Use the selected securities to create the MILP problem.

5. *Local MILP problem solution*: Solve the instance of the problem created at the previous step and indicate with x_I the vector of solutions. Let s be the number of securities with value different from zero in x_I and $k-s$ the components of x_I with value equal to zero, sorted in the nondecreasing order of the reduced cost coefficients. Replace the last $\lfloor (k-s)/2 \rfloor$ securities with value zero with the first $\lfloor (k-s)/2 \rfloor$ securities of the vector x_R. Cancel the first $\lfloor (k-s)/2 \rfloor$ components of x_R. Repeat steps 4 and 5 until a total limit number of securities N^* has been considered, keeping the best solution found.

A variant of this procedure can be obtained by modifying the number $(k-s)/2$ of securities which are replaced at each iteration or the stopping rule. For instance, a maximum number of iterations can be fixed.

5. Computational experiences in the Milan Stock Exchange

The mixed integer programming model for portfolio selection has been applied to data from the Milan Stock Exchange: all the considered investment alternatives are represented by securities quoted on the Milan Stock Exchange.

We briefly recall that in the Milan Stock Exchange the minimum transaction lot, directly fixed by the Stock Exchange Board for each security, is worked out by using as reference the number of securities for stocks and the nominal value for bonds and Government securities.

Table 1 reports the percent weight of taxes on the total amount of money invested. Taxes represent an additional proportional cost which varies according to the type of traders involved and the kind of operation realized.

The three groups of operations A, B and C differ in their objective indicating respectively stakes in all the types of company (group A), money values (group B) and Government securities and bonds (group C). The groups referred to as 1, 2 and 3 indicate the different kinds of intermediaries involved in the agreement: agreements set among brokers and specialized institutes, the so-called Italian SIM (first group), agreements among banks or other institutional traders and private buyers (second group), agreements realized among noninstitutional traders (third group).

According to this distinction, it is easy to work out the average commission. For example, if we consider a private investor who decides to put his money in a portfolio of stocks his average commission will be of 0.5% while his taxation on the agreement will be equal to 0.05% (see cell $(A,2)$ of Table 1) for a total proportional cost of 0.55%. Since we have supposed to operate exactly in this condition, we have set $d_j = 0.0055$, $\forall\, j \in S$.

The computational experiences have been carried out on two sets of data which cover the disjoint time periods 1989–1991 and 1992–1994 with a total of 277 securities available in the first set and 244 in the second one. From now on, we define by S_1 the time period (1989–1991) and by S_2 the time period (1992–1994).

Securities quotation is not always available on the market. The availability of securities price in

Table 1
Percent weight of taxation over amount invested

Operations	Traders		
	1 (%)	2 (%)	3 (%)
A	0.012	0.05	0.14
B	0.04	0.09	0.10
C	0.009	0.009	0.016

the market, at a given date, is related to the way in which the securities are admitted to quotation. Due to technical reasons the Exchange Board can decide to exclude from quotation on the market some securities for a limited time period (days or weeks) during which the security itself or the corresponding issuing company do not satisfy fixed acceptability standards. The two selected groups of securities include all the securities which have never been excluded from quotation respectively in the time periods S_1 and S_2.

We have created 48 different problem instances for each time period. The instances have been formulated according to different levels of capital and to different values for the parameter ρ. Specifically, the fund C which represents the amount of money that, at the end, will result to be invested in the portfolio, is allowed to range within the interval (C_0, C_1). We have established four different values for C_0 (100, 500, 1000, 5000 million of Italian Liras) and we have fixed C_1 (the upper bound) as follows:

$$C_1 = C_0 + \gamma\ C_0. \tag{22}$$

In this way C_1 is carried out by increasing C_0 (the lower bound) by a fixed percentage γ. Two values have been considered for γ (1% and 2% respectively) for a total number of capital intervals equal to eight.

Efficient frontiers, generated for each capital range in each time period, are analyzed with respect to six different values for the expected return: the parameter ρ is supposed to range between a minimum value of 0.25% per month to a maximum of 1.50%, with a step of 0.25%, taking the values 0.25, 0.50, 0.75, 1.00, 1.25 and 1.50 respectively.

We tried to solve each instance by using CPLEX, on a Personal Computer 52E with microprocessor Intel Pentium (120 MHz). We fixed a treememory parameter to 50Mb of memory. This parameter sets an upper limit on the amount of memory that the branch-and-bound tree can use. This implies that CPLEX will terminate when the amount of memory required to store branch-and-bound tree information exceeds the treememory parameter setting. With this memory limit, the computational time ranged between few minutes to a maximum of 30 minutes. In Table 2, a '*' shows the instances in which the optimum was not found within the memory limit. For the solved instances the number of positive variables, that is the number of selected securities, is shown. In each cell the first number refers to time period S_1 while the second one to S_2. Note that this number is always below 20 and almost always below 15. It is worth noticing that when the capital range is (100–101) and (100–102) million of Italian Liras, the MILP is always solved to optimality in both time periods, while for all the other cases this is not always possible. For the instances in which the capital varies in the intervals (500–505) and (500–510) the problem is always solved to optimality when data belonged to the period S_1, while it is solved to optimality five times over six for the period S_2: in this last case the unique critical value is for $\rho = 1.50$ corresponding to a 19.56% a year rate of return. As far as period S_2 is concerned, for the cases with capital ranging within (1000–1010) or within (1000–1020) no integer optimum is found

Table 2
Number of positive variables in the optimal solutions for the time periods S_1 and S_2

Capital range	Expected return					
	$\rho = 0.25$	$\rho = 0.50$	$\rho = 0.75$	$\rho = 1.00$	$\rho = 1.25$	$\rho = 1.50$
100–101	8–12	6–10	5–10	6–10	7–8	7–8
100–102	6–12	6–10	5–10	6–10	7–8	7–8
500–505	12–15	12–15	11–14	9–14	10–14	9–*
500–510	12–15	12–15	11–14	9–14	10–14	9–*
1000–1010	*–17	*–16	*–17	8–*	10–15	11–15
1000–1020	*–17	*–16	*–17	8–*	10–15	11–15
5000–5050	*–19	*–18	*–18	*–*	*–18	*–*
5000–5100	*–19	*–18	*–18	*–*	*–18	11–*

for $\rho = 1.00$, while for the instances with capital range equal to (5000–5050) or to (5000–5100) the integer optimum is reached only four times over six (for $\rho = 0.25$, $\rho = 0.50$, $\rho = 0.75$ and $\rho = 1.25$). Finally, no optimal solution is determined for the period S_1 when the capital ranges between 5000 and 5100 with the lone exception for $\rho = 1.50$, while for the instances with (1000–1010) or (1000–1020) only three cases over six, those with the highest rates of return, are successful.

Whenever the optimum was found within this limit, the error generated by each heuristic is evaluated as

$$\frac{z^H - z^*}{z^*}, \tag{23}$$

where z^H and z^* are the values of the objective function found by the heuristic and the optimal value. In the cases when the optimum was not available, the error of the heuristic is evaluated with respect to the optimal solution of the relaxed problem z^R as

$$\frac{z^H - z^R}{z^R}. \tag{24}$$

Note that the latter errors represent an overestimation of the errors with respect to the optimum.

In each table which shows the computational results of the heuristics the symbol '*' identifies the errors computed with respect to the relaxed optimal solution. Herewith, we present in detail the computational results. The groups Tables 3–5 and Tables 6–8 show the percent errors reported on each instance (defined by a fixed range of capital and a fixed rate of return) by procedure A (the

Table 3
Percent errors for the Basic MILP-based heuristic – Time period S_1

Capital range	Expected return						
	$\rho = 0.25$(%)	$\rho = 0.50$(%)	$\rho = 0.75$(%)	$\rho = 1.00$(%)	$\rho = 1.25$(%)	$\rho = 1.50$(%)	Average(%)
100–101	0.897	5.588	5.252	3.430	1.028	0.978	2.862
100–102	2.669	5.326	4.032	3.430	1.028	0.978	2.910
500–505	0.892	0.211	1.451	0.097	0.212	0.0263	0.482
500–510	0.892	0.211	1.451	0.097	0.212	0.0263	0.482
1000–1010	2.189*	1.061*	1.895*	0.00	0.0351	0.141	0.887
1000–1020	2.189*	1.061*	1.895*	0.00	0.0351	0.141	0.887
5000–5050	0.189*	0.281*	0.318*	0.114*	0.067*	0.06*	0.172
5000–5100	0.189*	0.281*	0.318*	0.114*	0.067*	0.009	0.163
Average	1.263	1.753	2.077	0.91	0.336	0.295	1.106

Table 4
Percent errors for the Reduced cost MILP-based heuristic $B(a)$, with $k = 2n$ – Time period S_1

Capital range	Expected return						
	$\rho = 0.25$(%)	$\rho = 0.50$(%)	$\rho = 0.75$(%)	$\rho = 1.00$(%)	$\rho = 1.25$(%)	$\rho = 1.50$(%)	Average(%)
100–101	0.00	0.00	0.00	0.00	0.00	0.00	0.00
100–102	0.00	0.00	0.00	0.00	0.00	0.00	0.00
500–505	0.101	0.00	0.00	0.00	0.00	0.00	0.0168
500–510	0.101	0.00	0.00	0.00	0.00	0.00	0.0168
1000–1010	1.604*	0.902*	0.872*	0.00	0.00	0.00	0.563
1000–1020	1.604*	0.902*	0.872*	0.00	0.00	0.00	0.563
5000–5050	0.171*	0.281*	0.122*	0.114*	0.0558*	0.05*	0.132
5000–5100	0.171*	0.281*	0.122*	0.114*	0.0558*	0.00	0.124
Average	0.469	0.296	0.248	0.028	0.014	0.006	0.177

Table 5
Percent errors for the Iterated MILP-based heuristic – Time period S_1

Capital range	Expected Return						
	$\rho = 0.25\%$	$\rho = 0.50\%$	$\rho = 0.75\%$	$\rho = 1.00\%$	$\rho = 1.25\%$	$\rho = 1.50\%$	Average %
100–101	0.00	0.00	0.00	0.00	0.00	0.00	0.00
100–102	0.00	0.00	0.00	0.00	0.00	0.00	0.00
500–505	0.00	0.00	0.00	0.00	0.00	0.00	0.00
500–510	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1000–1010	1.472*	0.902*	0.599*	0.00	0.00	0.00	0.495
1000–1020	1.472*	0.902*	0.599*	0.00	0.00	0.00	0.495
5000–5050	0.171*	0.195*	0.122*	0.094*	0.0558*	0.05*	0.115
5000–5100	0.171*	0.195*	0.122*	0.094*	0.0558*	0.00	0.106
Average	0.411	0.274	0.180	0.0235	0.014	0.006	0.151

Table 6
Percent errors for the Basic MILP-based heuristic – Time period S_2

Capital range	Expected return						
	$\rho = 0.25(\%)$	$\rho = 0.50(\%)$	$\rho = 0.75(\%)$	$\rho = 1.00(\%)$	$\rho = 1.25(\%)$	$\rho = 1.50(\%)$	Average(%)
100–101	0.235	1.969	0.00	3.317	0.00	4.243	1.627
100–102	0.032	0.342	0.00	3.076	0.00	3.927	1.229
500–505	0.00	0.351	0.574	0.205	0.090	1.448*	0.445
500–510	0.00	0.351	0.574	0.205	0.090	1.448*	0.445
1000–1010	0.082	0.0053	0.308	0.942*	0.00	0.087	0.237
1000–1020	0.082	0.0053	0.308	0.942*	0.00	0.087	0.237
5000–5050	0.117	0.0462	0.0443	0.153*	0.037	1.421*	0.303
5000–5100	0.117	0.0462	0.0443	0.153*	0.037	1.421*	0.303
Average	0.0835	0.389	0.232	1.124	0.027	1.760	0.603

Table 7
Percent errors for the Reduced Cost MILP-based heuristic $B(a)$, with $k = 2n$ – Time period S_2

Capital range	Expected return						
	$\rho = 0.25(\%)$	$\rho = 0.50(\%)$	$\rho = 0.75(\%)$	$\rho = 1.00(\%)$	$\rho = 1.25(\%)$	$\rho = 1.50(\%)$	Average(%)
100–101	0.00	0.00	0.00	0.00	0.00	2.615	0.436
100–102	0.00	0.00	0.00	0.00	0.00	2.210	0.368
500–505	0.00	0.0734	0.00	0.00	0.00	1.338*	0.235
500–510	0.00	0.0734	0.00	0.00	0.00	1.338*	0.235
1000–1010	0.00	0.00	0.00	0.904*	0.00	0.00	0.151
1000–1020	0.00	0.00	0.00	0.904*	0.00	0.00	0.151
5000–5050	0.00	0.00	0.00	0.153*	0.037	0.926*	0.186
5000–5100	0.00	0.00	0.00	0.153*	0.037	0.926*	0.186
Average	0.00	0.0184	0.00	0.264	0.009	1.169	0.243

Basic MILP-based heuristic), by procedure B (we specifically present the results for the first version (rule (a) of step 3) of the procedure fixing $k = 2n$, i.e. the case in which the value k is fixed to be equal to two times the number n of positive components of the solution of the relaxed problem) and by procedure C (the Iterated MILP-based heuristic is implemented using as securities selection rule $k = 2n$ while the stopping rule has been fixed to $N^* = 100$, as suggested by the results obtained for

Table 8
Percent errors for the Iterated MILP-based heuristic – Time period S_2

Capital range	Expected return						
	$\rho = 0.25$(%)	$\rho = 0.50$(%)	$\rho = 0.75$(%)	$\rho = 1.00$(%)	$\rho = 1.25$(%)	$\rho = 1.50$(%)	Average(%)
100–101	0.00	0.00	0.00	0.00	0.00	0.00	0.00
100–102	0.00	0.00	0.00	0.00	0.00	0.00	0.00
500–505	0.00	0.00	0.00	0.00	0.00	1.323*	0.2205
500–510	0.00	0.00	0.00	0.00	0.00	1.323*	0.2205
1000–1010	0.00	0.00	0.00	0.904*	0.00	0.00	0.151
1000–1020	0.00	0.00	0.00	0.904*	0.00	0.00	0.151
5000–5050	0.00	0.00	0.00	0.153*	0.00	0.121*	0.0457
5000–5100	0.00	0.00	0.00	0.153*	0.00	0.121*	0.0457
Average	0.00	0.00	0.00	0.264	0.00	0.361	0.104

different values of N^* in Mansini [15]), for the time periods S_1 and S_2, respectively.

It is evident how, in some instances, all the heuristic procedures terminate with the same objective function value. For example, with capital range (500–505) and expected rate of return equal to 0.25% per month all the heuristics get the optimal solution over the period S_2. However, while CPLEX takes about 6 min to get the integer optimum, the Basic MILP-based heuristic less than 20 s, the Reduced cost MILP-based heuristic about 1 min and the Iterated MILP-based heuristic about 5 min. Similarly, if we consider the case (1000–1020) for the capital and a higher rate of return, for example $\rho = 1.25\%$ (i.e. a rate of return greater than 12% per year) the computational time of both CPLEX and the heuristics increase and so does the gap between the performances. While CPLEX is unable to find the optimum within 30 min, the Basic MILP-based heuristic takes only 2 min and 45 s and the Iterated MILP-based heuristic 9 min and 29 s.

We can notice from Table 9 that the computational time is not directly depending from the number of securities. If on an average the computational time increases with the number of securities, we observe that procedure A solved with respect to S_1 has an average time lower than procedure A with respect to S_2. The result is partially justified by the fact that in some instances the number of securities selected in the optimal solution is lower in S_1 than in S_2.

As it can be easily verified looking at the tables, the errors produced by procedure B are smaller than the errors produced by procedure A and the errors of procedure C are smaller than the errors of procedure B. However, smaller errors imply higher computational time. In particular, the Iterated MILP-based heuristic reaches the optimal integer solution on 40 over 48 different considered instances of the problem when the period S_2 is taken into account, while the ratio decreases to 31 over 48 when the period S_1 is considered. The only instances in which a positive error is shown are those for which the optimum was not found. In these cases the errors shown in the tables for the iterated procedure, always smaller than 1.472% for S_1 and smaller than 1.323% for S_2, may be due to the gap between the optimum and the relaxed solution. The number of times procedure A gets the integer optimum is 8 over 48 for S_2 and only 2 over 48 for S_1, with a maximum error equal to

Table 9
Minimum, average and maximum computational times (in minutes and seconds) for the heuristic procedures

	1989–1991 (277)			1992–1994 (244)		
	Min	Average	Max	Min	Average	Max
Proced. A	17″	38″	2′20″	24″	51″	3′20″
Proced. B	27″	1′53″	7′3″	36″	1′34″	6′21″
Proced. C	2′20″	11′21″	17′13″	4′31″	10′6″	19′14″

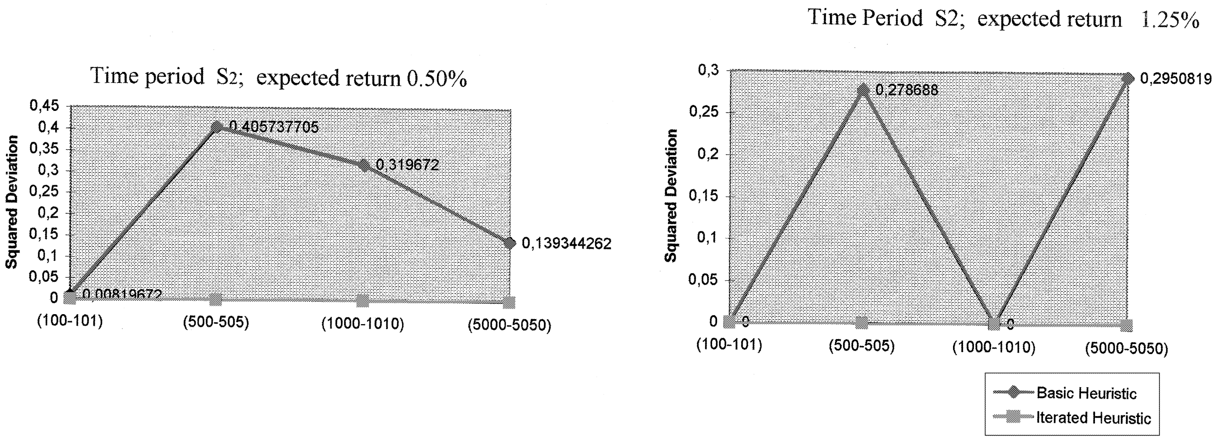

Fig. 1.

4.243% in the first case and equal to 5.588% in the second one. However, while the average computational time required by the Basic MILP-based heuristic does not exceed 38 s for S_1 and 51 s for S_2, the Iterated MILP-based heuristic has an average computational time greater than 11 min for the period S_1 and 10 min for the period S_2, reaching a peak of more than 17 min in some instances of S_1 and of 19 min in the other time period.

From the above results, it is evident that the proposed heuristics, and especially the last one, are extremely effective. It is also interesting to see the differences in portfolio composition between the optimal portfolio and the portfolios obtained by the heuristics. In Fig. 1 we show such differences, as functions of budget size, for the time period S_2 and with monthly rates of return $\rho = 0.50\%$ and $\rho = 1.25\%$. We used as measure of the difference in composition the quantity

$$\frac{\sum_{i \in S}(x_i^* - x_i^{\mathrm{H}})^2}{|S|},$$

where x_i^* and x_i^{H} denote the number of units of security i in the optimal and in the heuristic solutions, respectively. While Procedure C always finds the optimal portfolio in the considered instances, Procedure B does not obtain the optimal portfolio in two cases, namely for the capital range (500,505) when $\rho = 0.50\%$ and for the capital range (5000,5050) when $\rho = 1.25\%$.

In order to give an idea of the detailed portfolio composition and of the actual size of the minimum transaction lots in the Italian Stock Exchange, in Table 10 we present more detailed results for one experiment over the time period 1992–1994 with an expected monthly rate of return equal to 0.75% and the capital ranging from 1000 to 1010 million Italian Liras. The values are expressed in thousands of Italian Liras. The first two columns report the securities selected and the corresponding value for the minimum transaction lot, respectively. The third column shows the optimal integer solution: we notice that in this case the integer optimal solution is also found by the heuristics B and C (see Tables 7 and 8). In the last column the actual investment for each security is shown.

Finally, Tables 11 and 12 show the results obtained for the heuristic algorithm proposed by Speranza [12]. The heuristic, which is inspired by the natural behavior of an investor which solves the relaxed problem and then adapts the solution to the integrality constraints, does not require the solution of any MILP problem. However, the results show that the heuristic never reaches the optimum and may generate quite large errors, with an average of 23% over the period (1992–1994). Moreover the algorithm may terminate without a feasible solution (see first line of Tables 11 and 12).

As it is common practice to solve the portfolio selection problems with minimum lots by applying a rounding procedure to the relaxed optimal so-

Table 10
Portfolio composition for the case with $C_0 = 1000$ and $C_1 = 1010$ million Italian Liras and $\rho = 0.75$. Time period S_2

Security	Min. trans. lot	Solution	Investment
Finarte prv	2375	9	21,375
Finarte ord.	5050	3	15,150
Gewiss	11,450.5	1	11,450.5
Riva Finanziaria	4600	11	50,600
Bonifiche Ferraresi	6696	1	6696
Merloni ord.	9435	10	94,350
Teknecomp ord.	9880	1	9880
Bayer	11,750	9	105,750
Volkswagen	13,750	12	165,000
Cantoni Itc ord.	5750	7	40,250
Mondadori ord.	6779.5	2	13,559
Ausiliare	3986	70	279,020
Acquedotto De Ferrari ord.	4772	9	42,948
Marangoni	13,697.5	3	41,092.5
Terme Acqui ord.	2375	3	7125
Parmalat Finanziaria	7520	1	7520
Sme	4080	22	89,760

Table 11
Percent errors for Speranza's heuristic. Time period S_1

Capital range	Expected return						
	$\rho = 0.25\%$	$\rho = 0.50\%$	$\rho = 0.75\%$	$\rho = 1.00\%$	$\rho = 1.25\%$	$\rho = 1.50\%$	Average %
100–101	5.614	infeas.	infeas.	infeas.	infeas.	infeas.	5.614
100–102	7.469	24.095	infeas.	infeas.	7.493	infeas	13.019
500–505	3.636	3.881	3.258	5.037	3.431	1.128	3.395
500–510	3.636	3.881	3.258	5.037	5.328	1.128	3.711
1000–1010	3.546*	1.462*	2.713*	3.294	2.657	0.782	2.409
1000–1020	3.546*	1.462*	2.713*	3.294	2.657	0.782	2.409
5000–5050	2.468*	1.871*	1.224*	0.294*	1.183*	0.274*	1.219
5000–5100	2.468*	1.871*	1.224*	0.294*	0.237*	0.224	1.053
Average	4.048	5.503	2.398	2.875	3.284	0.719	3.25

Table 12
Percent errors for Speranza's heuristic. Time period S_2

Capital range	Expected return						
	$\rho = 0.25\%$	$\rho = 0.50\%$	$\rho = 0.75\%$	$\rho = 1.00\%$	$\rho = 1.25\%$	$\rho = 1.50\%$	Average %
100–101	1.890	infeas.	infeas.	infeas.	44.924	15.725	20.846
100–102	3.697	62.269	20.928	12.988	44.924	15.725	26.755
500–505	224.4	6.592	4.849	5.817	11.030	6.869*	43.26
500–510	224.4	6.592	4.849	5.901	11.030	7.267*	43.34
1000–1010	107	1.990	0.093	3.830*	5.571	2.704	20.198
1000–1020	107	1.990	0.093	3.830*	5.571	3.096	20.263
5000–5050	18.849	0.616	0.512	0.756*	3.495	0.528*	4.126
5000–5100	18.849	0.616	0.512	0.756*	3.495	0.528*	4.126
Average	88.26	11.52	4.548	4.84	16.255	6.555	23

Table 13
Relative errors for Speranza's heuristic with respect to the Iterated MILP-based Heuristic. Time period S_1

Capital range	Expected return						
	$\rho = 0.25\%$	$\rho = 0.50\%$	$\rho = 0.75\%$	$\rho = 1.00\%$	$\rho = 1.25\%$	$\rho = 1.50\%$	Average %
100–101	5.614	–	–	–	–	–	5.614
100–102	7.469	24.095	–	–	7.493	–	13.019
500–505	3.636	3.881	3.258	5.037	3.431	1.128	3.395
500–510	3.636	3.881	3.258	5.037	5.328	1.128	3.711
1000–1010	2.044	0.555	2.101	3.294	2.657	0.782	1.905
1000–1020	2.044	0.555	2.101	3.294	2.657	0.782	1.905
5000–5050	2.293	1.673	1.101	0.199	1.126	0.2338	1.104
5000–5100	2.293	1.673	1.101	0.199	1.126	0.224	1.103
Average	3.629	5.187	2.153	2.843	3.403	0.713	3.085

Table 14
Relative errors for Speranza's heuristic with respect to the Iterated MILP-based Heuristic. Time period S_2

Capital range	Expected return						
	$\rho = 0.25\%$	$\rho = 0.50\%$	$\rho = 0.75\%$	$\rho = 1.00\%$	$\rho = 1.25\%$	$\rho = 1.50\%$	Average %
100–101	1.890	–	–	–	44.924	15.725	20.846
100–102	3.697	62.269	20.928	12.988	44.924	15.725	26.755
500–505	224.4	6.592	4.849	5.817	11.030	5.473	43.027
500–510	224.4	6.592	4.849	5.901	11.030	5.866	43.106
1000–1010	107	1.990	0.093	2.899	5.571	2.704	20.043
1000–1020	107	1.990	0.093	2.899	5.571	3.096	20.108
5000–5050	18.849	0.616	0.512	0.602	3.495	0.406	4.08
5000–5100	18.849	0.616	0.512	0.602	3.495	0.406	4.08
Average	88.26	11.52	4.548	4.530	16.255	6.175	22.805

lution, it is interesting to analyze the relative efficiency gain of our heuristics with respect to a rounding procedure. At this aim, in Tables 13 and 14 we give the relative errors generated by the rounding procedure proposed by Speranza with respect to the best of our heuristics, namely the Iterated MILP-based Heuristic. When the rounding procedure has been unable to find any feasible solution a '–' appears. It can be observed that the rounding procedure never happens to find a better solution than the Iterated heuristic. Moreover, especially on S_2, our heuristic greatly reduces the errors produced by the rounding procedure.

6. Conclusions and future research

In this paper the model for portfolio selection with minimum rounds has been applied, by using three different heuristics, to the Milan Stock Exchange. The experiments have shown how the high computational complexity of the problem may prevent from getting the optimal solution in a reasonable amount of time, while the proposed heuristics get very good solutions in a reasonable computational time. The average error generated by the Iterated MILP-based procedure, the most effective heuristic, is 0.151% with a maximum error of 1.472% for the time period S_1 and 0.104%, with a maximum error of 1.323% for S_2. The proposed heuristic which requires the shortest computational time generates an average error of 0.603% in S_2 and of 1.106% in S_1, with a maximum of 4.243% in the first case and of 5.588% in the second one.

Among the possible future directions of research, on one side it is of interest to evaluate the performance of the proposed heuristics when applied to the variant of Markowitz's model which takes the rounds into account, although some problems may arise with the solution of a quadratic model with integer variables. On the other hand, the present model, opportunely changed, can be used to manage a selection portfolio problem based on derivatives, which implies a higher difficulty of risk management due to their asymmetry.

Acknowledgements

We are grateful to two anonymous referees for the suggestions which helped us to improve the presentation of the paper.

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